ISRAEL JOURNAL OF MATHEMATICS, Vol. 47, Nos. 2-3, 1984

UNIFORMLY ERGODIC MAPS ON C^*-ALGEBRAS

BY

ULRICH GROH

ABSTRACT

Let T be an identity preserving Schwarz map on a C^*-algebra. The following conditions are proved to be equivalent: (a) *T is uniformly ergodic with finite-dimensional fixed space.* (b) *T is quasi-compact.*

1. Let E be a Banach space. An operator $T \in \mathscr{L}(E)$ is called *uniformly ergodic* (resp. *strongly ergodic*) if the averages

$$T_n := n^{-1} \sum_{k=0}^{n-1} T^k \qquad (n \in \mathbf{N})$$

converge in the uniform operator topology (resp. strong operator topology). The limit P of the sequence (T_n) is called the ergodic projection associated with T and satisfies

$$P = P^2 = PT = TP.$$

It follows that P is a projection onto the fixed space

$$F(T) := \{x \in E : Tx = x\}.$$

An operator $T \in \mathscr{L}(E)$ is called *quasi-compact* if there exists a compact operator K on E and a natural number m such that

$$\| T^m - K \| < 1.$$

Under the assumption that $(n^{-1}\| T^n \|)$ converges to zero T is quasi-compact iff the peripheral spectrum $\sigma(T) \cap \Gamma$ of T contains only poles of the resolvent $R(\lambda, T)$ of first order with finite-dimensional eigenspaces, whereas T is uniformly ergodic iff 1 is a pole of the resolvent of order at most one ([2, App. W.],

Received March 1, 1982 and in revised form December 15, 1983

[3, VIII. 8]) or iff $(I-T)E$ is closed ([7]). Thus quasi-compactness implies uniform ergodicity but not conversely, even if the fixed space of T is finite-dimensional.

If a C^*-algebra is commutative it is a Banach lattice ([14]) and, by [9], a positive contraction with $F(T)$ finite-dimensional is quasi-compact if and only if it is uniformly ergodic. Since a non-commutative C^*-algebra is not a Banach lattice, the aim of this paper is to extend this surprising result to arbitrary C^*-algebras.

For the definitions and notations concerning the *spectrum* $\sigma(T)$, *resolvent set* $\rho(T)$, *spectral radius* $r(T)$, *resolvent* $R(\lambda, T)$, *pole of the resolvent*, etc., of a bounded operator T on a (complex) Banach space E we refer to Dunford–Schwartz [3]. We recall specifically that the *approximate point spectrum* $A\sigma(T)$ is the set of all $\alpha \in \mathbf{C}$ for which there exists a normalized sequence (x_n) in E such that $\lim_n \|(\alpha - T)x_n\| = 0$. In particular, the *point spectrum* $P\sigma(T)$ and the *peripheral spectrum* $\sigma(T) \cap r(T)\Gamma$ are contained in $A\sigma(T)$, where Γ is the set of all complex numbers of modulus one.

The C^*-algebras under consideration are always unital. If $\mathfrak{A}$ is a C^*-algebra with positive cone $\mathfrak{A}_+ := \{x^*x : x \in \mathfrak{A}\}$, a map $T \in \mathscr{L}(\mathfrak{A})$ is called *positive*, if $T(\mathfrak{A}_+) \subseteq \mathfrak{A}_+$ and is called a *Schwarz* map, if $T(x)T(x)^* \leqq \|T\| T(xx^*)$ for every x in $\mathfrak{A}$. Recall that every positive map on a commutative C^*-algebra is a Schwarz map (see, e.g., [14, IV.3.9]).

2. The main result of this section concerns the eigenspaces pertaining to peripheral eigenvalues of identity preserving Schwarz maps with finite-dimensional fixed-spaces. Let us first recall some facts about w*-continuous linear forms on a W*-algebra $\mathfrak{A}$. Let $\varphi \in \mathfrak{A}_*$, $\mathfrak{A}_*$ the predual of $\mathfrak{A}$. Then there exists a positive linear form $|\varphi| \in \mathfrak{A}_*$ and a partial isometry $u \in \mathfrak{A}$ uniquely determined by the conditions

$$\varphi(x) = |\varphi|(xu) =: (R_u|\varphi|)(x) \qquad (x \in \mathfrak{A})$$

$$u^*u = s(|\varphi|)$$

where $s(|\varphi|)$ is the support projection of $|\varphi|$. We refer to this as the *polar decomposition* of φ ([13, 5.16]). For the polar decomposition of φ^*, where $\varphi^*(x) = \varphi(x^*)^*$ $(x \in \mathfrak{A})$, we obtain

$$\varphi^* = R_{u^*}|\varphi^*|, \qquad |\varphi^*| = L_{u^*}R_u|\varphi|, \qquad uu^* = s(|\varphi^*|)$$

([13, E.5.10]), where for $a \in \mathfrak{A}$ the maps R_a (resp. L_a) are given by $(x \mapsto xa)$ (resp. $(x \mapsto ax)$). In addition $|\varphi|$ is uniquely determined by the properties

$$\|\varphi\| = \| |\varphi| \|$$

$$|\varphi(x)|^2 \leqq \|\varphi\| \, |\varphi| (xx^*) \qquad (x \in \mathfrak{A})$$

([13, E.5.11], [14, III.4.6])

2.1. PROPOSITION. *Let $\mathfrak{A}$ be a W^*-algebra, let $T \in \mathscr{L}(\mathfrak{A})$ be an identity preserving Schwarz map with preadjoint $T_* \in \mathscr{L}(\mathfrak{A}_*)$, let α be a peripheral eigenvalue of T_* and let $\varphi \in \ker(\alpha - T_*)$ be of norm one with polar decomposition $\varphi = R_u |\varphi|$. Then the following assertions hold*:

(a) *$|\varphi|$ and $|\varphi^*|$ are elements of $F(T_*)$.*

(b) *Suppose $T(s(|\varphi|)\mathfrak{A}) \subseteq s(|\varphi|)\mathfrak{A}$ and suppose there exists a faithful family Ψ of T_*-invariant states on $\mathfrak{A}$. Then $(\alpha^* T)(x) = u^* T(ux) = (L_{u^*} \circ T \circ L_u)(x)$ for all $x \in s(|\varphi|\mathfrak{M})$ and $s(|\varphi^*|)\mathfrak{M}$ is T-invariant.*

PROOF. (a) Using the Schwarz inequality for T we obtain for $x \in \mathfrak{A}$:

$$|\varphi(x)|^2 = |\varphi(Tx)|^2 \leqq |\varphi|((Tx)(Tx)^*) \leqq (T_*|\varphi|)(xx^*).$$

Because $\|\varphi\| = |\varphi|(1) = |\varphi|(T1) = \|T_*|\varphi|\|$, it follows that $|\varphi| \in F(T_*)$ by the characterization mentioned above.

(b) Suppose $T(s(|\varphi|)\mathfrak{A}) \subseteq s(|\varphi|)\mathfrak{A}$. We show first $Tu^* = \alpha u^*$. Using the Schwarz inequality we obtain

$$(Tu^* - \alpha u^*)(Tu^* - \alpha u^*)^* \leqq T(u^*u) + u^*u - \alpha u^*(Tu) - \alpha^*(Tu^*)u.$$

Because $|\varphi| \in F(T_*)$, $\varphi \in \ker(\alpha - T_*)$ and $\varphi^* \in \ker(\alpha^* - T_*)$ it follows that

$$\begin{aligned} &|\varphi|((Tu^* - \alpha u^*)(Tu^* - \alpha u^*)^*) \\ &\leqq 2|\varphi|(u^*u) - \alpha|\varphi|(u^*(Tu)) - \alpha^*|\varphi|((Tu^*)u) \\ &= 2|\varphi|(u^*u) - \varphi^*(u) - \varphi(u^*) \\ &= 2|\varphi|(u^*u) - 2|\varphi|(u^*u) \\ &= 0. \end{aligned}$$

Since $u^* \in s(|\varphi|)\mathfrak{A}$,

$$0 \leqq (Tu^* - \alpha u^*)(Tu^* - \alpha u^*)^* \in s(|\varphi|)\mathfrak{A}s(|\varphi|),$$

and because $|\varphi|$ is faithful on the W^*-algebra $s(|\varphi|)\mathfrak{A}s(|\varphi|)$, it follows that

$$(Tu^* - \alpha u^*)(Tu^* - \alpha u^*)^* = 0$$

which implies $Tu^* = \alpha u^*$.

For $x, y \in \mathfrak{A}$ define
$$B(x,y) = T(xy^*) - T(x)T(y)^*.$$
Then $B(\cdot,\cdot)$ is a sesquilinear, positive map from $\mathfrak{A}\times\mathfrak{A}$ in $\mathfrak{A}$ satisfying
$$B(x,x)=0 \quad \text{iff } B(x,y)=0 \qquad \text{for all } y \in \mathfrak{A}.$$

To prove this one has only to note that $\psi \circ B$ is a positive, hermitian sesquilinear form on $\mathfrak{A}$ for every $\psi \in \mathfrak{A}_+^*$. Hence $\psi(B(x,x))=0$ iff $\psi(B(x,y))=0$ for all $y \in \mathfrak{A}$. Since $\mathfrak{A}_+^*$ is generating, the assertion follows.

Because of $T(uu^*) \geqq uu^*$ and because the family Ψ is faithful and consists of T_*-invariant states, it follows that $T(uu^*) = uu^*$ and similarly $T_*((uu^*)^2) = (uu^*)^2$. Therefore $B(u,x)=0$ and $B(uu^*,x)=0$ for all $x \in \mathfrak{A}$. Thus $u^*T(ux) = \alpha^*T(x)$ for all $x \in s(|\varphi|)\mathfrak{A}$ since $s(|\varphi|)\mathfrak{A}$ is T-invariant, and $T(s(|\varphi^*|)\mathfrak{A}) \subseteq s(|\varphi^*|)\mathfrak{A}$. ∎

2.2. Theorem. *Let $\mathfrak{A}$ be a W*-algebra and let $T \in \mathcal{L}(\mathfrak{A})$ be an identity preserving Schwarz map with preadjoint $T_* \in \mathcal{L}(\mathfrak{A}_*)$. If the fixed space of T_* is finite-dimensional, then* $\dim\ker(\alpha - T_*) \leqq \dim\ker F(T_*)$ *for all $\alpha \in \mathbf{C}$ of modulus one.*

Proof. Let $\Psi := \{|\psi| : \psi \in F(T_*)\}$ and let us first assume Ψ to be faithful. Since then T_* is strongly ergodic on $\mathfrak{A}_*$ ([6]), $\dim F(T_*) = \dim F(T) < \infty$.

Using the sesquilinear mapping $B(\cdot,\cdot)$ introduced in the proof of Proposition 2.1(b), it is easy to see that $F(T)$ is a finite-dimensional C^*-subalgebra of $\mathfrak{A}$. Hence we can decompose $F(T)$ into the direct sum
$$F(T) = \bigoplus_{k=1}^{m} M_k$$
where each M_k is isomorphic to the C^*-algebra of all $n_k \times n_k$-matrices and where the sequence $\{n_1, \cdots, n_m\}$ of positive integers is uniquely determined by $F(T)$ (up to a permutation) ([14, I.11.2]). Thus there exists a sequence $\{p_1, \cdots, p_r\}$ of mutually orthogonal, minimal projections in $F(T)$ such that $r = \sum_{i=1}^m n_i$ and $\sum_{j=1}^r p_j = 1$. Since $F(T)$ is a C^*-subalgebra of $\mathfrak{A}$ these projections are mutually orthogonal in $\mathfrak{A}$ with sum 1.

Because of $T(p_j) = p_j$ $(1 \leqq j \leqq r)$ it follows that $T(p_jx) = p_jT(x)$ for all $x \in \mathfrak{A}$. Thus the w*-closed right ideal $\mathfrak{R}_j := p_j\mathfrak{A}$ is T-minimal, i.e., if $\mathfrak{R} \subseteq \mathfrak{R}_j$ is another T-invariant w*-closed right ideal then $\mathfrak{R}$ is equal to $\{0\}$ or $\mathfrak{R}_j$.

For $j \in \{k : 1 \leqq k \leqq r\}$ consider $\mathfrak{T}_j := \{L_{p_j}\psi : \psi \in \mathfrak{A}_*\}$ and note that $\mathfrak{A}_* = \bigoplus_{j=1}^r \mathfrak{T}_j$ and $T_*(\mathfrak{T}_j) \subseteq \mathfrak{T}_j$. If $\varphi \in \mathfrak{T}_j$ with polar decomposition $\varphi = R_u|\varphi|$, then it is easy to see that $|\varphi| \in \mathfrak{T}_j$ and $u^*u \leqq p_j$. In particular, $u^* \in \mathfrak{R}_j$. Let

$\alpha \in P\sigma(T_*) \cap \Gamma$; since $P\sigma(T_*) = \bigcup_{j=1}^m P\sigma(T_{|\mathfrak{T}_j})$ there exists a $j \in \{k : 1 \leqq k \leqq r\}$ and a normalized $\varphi \in \mathfrak{T}_j$ such that $T\varphi = \alpha\varphi$. Let

$$\mathfrak{N}_{|\varphi|} := \{x \in \mathfrak{R}_j : |\varphi|(xx^*) = 0\},$$

then $\mathfrak{N}_{|\varphi|}$ is a w*-closed right ideal contained in $\mathfrak{R}_j$, which is T-invariant, since $|\varphi| \in F(T_*)$ (Proposition 2.1(a)). Therefore $\mathfrak{N}_{|\varphi|} = \{0\}$, which implies $s(|\varphi|) = p_j$. By Proposition 2.1(b)

$$(*) \qquad (\alpha^* T)(x) = (L_{u^*} \circ T \circ L_u)(x)$$

for all $x \in \mathfrak{R}_j$. Since $s(|\varphi^*|)\mathfrak{A}$ is T-invariant (Proposition 2.1(b)), R_u is a bijection from $s(|\varphi|)\mathfrak{A}$ onto $s(|\varphi^*|)\mathfrak{A}$ with inverse R_{u^*} and because of

$$\dim F(T_{|s(|\varphi^*|)\mathfrak{A}}) \leqq \dim F(T)$$

it follows that

$$\dim \ker(\alpha - T_{|\mathfrak{R}_j}) \leqq \dim F(T)$$

for all $1 \leqq j \leqq r$. Since $\mathfrak{A} = \bigoplus_{j=1}^m \mathfrak{R}_j$ this implies

$$\dim \ker(\alpha - T) \leqq \dim F(T).$$

Because $\dim \ker(\alpha - T_*) \leqq \dim \ker(\alpha - T)$ (see, e.g., Proposition 3.1),

$$\dim \ker(\alpha - T_*) \leqq \dim F(T_*)$$

for all $\alpha \in \mathbf{C}$ of modulus one.

For the general case let $p := \sup\{s(|\psi|) : \psi \in F(T_*)\}$. Then $Tp \geqq p$ because $T(s(|\psi|)) \geqq s(|\psi|)$ for all $\psi \in F(T_*)$. We claim that the map T_p on $\mathfrak{A}_p := p\mathfrak{A}p$ given by

$$(x \mapsto p(Tx)p), \qquad x \in \mathfrak{A}_p$$

is well defined, is identity preserving and is a Schwarz map. In order to see that T_p is well defined note that for $y \in \mathfrak{A}$:

$$p(Ty - T(pyp)p) = p(T(y(1-p)) + T((1-p)yp)) = 0$$

because the right ideal $(1-p)\mathfrak{A}$ and the left ideal $\mathfrak{A}(1-p)$ are T-invariant. T_p is identity preserving because $T_p(p) \leqq p$ and $\Psi_{|p\mathfrak{A}_*p}$ is faithful. Since $T_p = Q \circ T \circ Q$ where Q is the Schwarz map $(x \mapsto pxp)$, T_p is a Schwarz map. If $\varphi \in p\mathfrak{A}_*p$ then for all $y \in \mathfrak{A}$

$$\varphi(Ty) = \varphi(p(Ty)p) = \varphi(p(T(pyp))p) = \varphi(T(pyp))$$

hence $T_*\varphi \in p\mathfrak{A}_*p$. The dual Banach space of $p\mathfrak{A}_*p$ is $\mathfrak{A}_p$. Therefore the adjoint of $T_{*|(p\mathfrak{A}_*p)}$ is T_p. If $\alpha \in \mathbf{C}$ is of modulus one and $\varphi_\alpha \in \ker(\alpha - T_*)$, then $\varphi_\alpha \in p\mathfrak{A}_*p$ because $s(|\varphi_\alpha|)$ and $s(|\varphi_\alpha^*|)$ are majorized by p. Thus $\ker(\alpha - T_*) \subseteq p\mathfrak{A}_*p$ for all such α. Since the family Ψ is faithful on the W*-algebra $\mathfrak{A}_p$ the theorem is proved. ∎

3. In this section we give a proof of our main theorem. First we need some preparations which we state separately. The result of our first proposition is well known ([15]), but for the convenience of the reader we give a (different) proof.

3.1. Proposition. *Let T be a contraction on a Banach space E with adjoint $T^* \in \mathscr{L}(E^*)$. Then for every peripheral eigenvalue α of T,* $\ker(\alpha - T^*)$ *separates the points of* $\ker(\alpha - T)$. *In particular,* $\dim\ker(\alpha - T) \leqq \dim\ker(\alpha - T^*)$.

Proof. Since for every $\alpha \in \mathbf{C}$ of modulus one $\ker(\alpha - T) = F(\alpha^* T)$ it suffices to prove that $F(T^*)$ separates the points of $F(T)$. Let $\mathfrak{U}$ be an ultrafilter on $[1,\infty)$ which converges to 1. Since the unit ball U^0 of E is $\sigma(E^*, E)$-compact and invariant under T^*, there exists for each $\psi \in U^0$

$$\psi_0 := \lim_{\mathfrak{U}} (\lambda - 1)R(\lambda, T^*)\psi.$$

Since T^* is $\sigma(E^*, E)$-continuous and

$$T^*R(\lambda, T^*) = \lambda R(\lambda, T^*) - I_{E^*}$$

we conclude $\psi_0 \in F(T^*)$. Now take $0 \neq x_0 \in F(T)$ and choose $\psi \in U^0$ such that $\psi(x_0) = 1$. From the considerations above it follows that

$$\psi_0(x_0) = \lim_{\mathfrak{U}} (\lambda - 1)\psi(R(\lambda, T)x_0) = \psi(x_0) = 1,$$

hence $0 \neq \psi_0 \in F(T^*)$ and $F(T^*)$ separates the points of $F(T)$. ∎

In the proof of our main theorem we use the so-called ultrapower $\hat{E}$ of a Banach space E with respect to a free ultrafilter $\mathfrak{U}$ on $\mathbf{N}$. For the construction of $\hat{E}$ we refer to [11, V.1.4]. Recall in particular that for $T \in \mathscr{L}(E)$ we have $\sigma(T) = \sigma(\hat{T})$ and $A\sigma(T) = A\sigma(\hat{T}) = P\sigma(\hat{T})$ where $\hat{T}$ is the canonical extension of T to $\hat{E}$.

3.2. Proposition. *Let E be a Banach space and $T \in \mathscr{L}(E)$. If $\alpha \in P\sigma(\hat{T})$ and* $\dim\ker(\alpha - \hat{T}) < \infty$, *then the following hold*:

(a) $\alpha \in P\sigma(T)$ *and* $\dim \ker(\alpha - T) = \dim \ker(\alpha - \hat{T})$.

(b) *If* T *is a contraction and* $|\alpha| = 1$ *then* α *is a pole of the resolvent* $R(\lambda, T)$.

PROOF. (a) Let $\alpha \in P\sigma(\hat{T})$ and suppose $\dim \ker(\alpha - \hat{T}) < \infty$. We prove first that $\alpha \in P\sigma(T)$. Let (x_n) be a normalized sequence in E such that $\lim_n \|(\alpha - T)x_n\| = 0$. Since $\alpha \in P\sigma(T)$ as soon as (x_n) has a convergent subsequence, we assume to the contrary that there is no such subsequence. Thus we may assume that there exists $\delta > 0$ such that $\|x_{n_1} - x_{n_2}\| \geqq \delta$ for all positive integers $n_1 \neq n_2$. For $i \in \mathbf{N}$ let $\hat{x}_i$ be the image of (x_{n+i}) in $\hat{E}$ and note that $\hat{x}_i \in \ker(\alpha - \hat{T})$ with $\|\hat{x}_i\| = 1$. Since $\ker(\alpha - \hat{T})$ is finite-dimensional the sequence $(\hat{x}_i)$ has an accumulation point in $\ker(\alpha - \hat{T})$. Thus there exist positive integers $j_1 < j_2$ such that

$$\|\hat{x}_{j_1} - \hat{x}_{j_2}\| \leqq \delta/2$$

which leads to a contradiction. Hence we proved $\alpha \in P\sigma(T)$.

Obviously $\dim \ker(\alpha - T) \leqq \dim \ker(\alpha - \hat{T})$. If

$$\dim \ker(\alpha - T) < \dim \ker(\alpha - \hat{T})$$

then there exist $\gamma > 0$ and a normalized $\hat{x} \in \ker(\alpha - \hat{T})$ with the property

$$\gamma \leqq \|\hat{x} - \hat{y}\| \qquad \text{for all } y \in \ker(\alpha - T).$$

Since every $(x_n) \in \hat{x}$ has a convergent subsequence by the result of the last paragraph, there exists $0 \neq z \in \ker(\alpha - T)$ such that

$$\gamma \leqq \|\hat{z} - \hat{y}\| \qquad \text{for all } y \in \ker(\alpha - T),$$

a contradiction. Thus $\dim \ker(\alpha - T) = \dim \ker(\alpha - \hat{T})$ as desired.

(b) Let $\dim \ker(\alpha - T) = n$ and choose $x_1, \cdots, x_n$ linearly independent of $\ker(\alpha - T)$. By Proposition 2.1 there exists $\varphi_1, \cdots, \varphi_n$ in $\ker(\alpha - T^*)$ such that $\varphi_i(x_j) = \delta_{i,j}$ $(i, j = 1, \cdots, n)$. Let $M := \bigcap_{i=1}^n \ker \varphi_i$, then $E = \ker(\alpha - T) \oplus M$ and $T(M) \subseteq M$. We claim $\alpha \notin \sigma(T_{|M})$; for if $\alpha \in \sigma(T_{|M})$, then there exists an approximative eigenvector sequence (y_n) in M pertaining to α. Since by the proof of (a) this sequence has a convergent subsequence, $M \cap \ker(\alpha - T) \neq \{0\}$, a contradiction. But α is a pole of the resolvent $R(\lambda, T_{|\ker(\alpha - T)})$, thus a pole of $R(\lambda, T)$. ■

3.3. THEOREM. *Let* T *be an identity preserving Schwarz map on a* C^**-algebra* $\mathfrak{A}$. *Then the following assertions are equivalent*:

(a) T *is uniformly ergodic with finite-dimensional fixed space.*

(b) T *is quasi-compact.*

(c) *The peripheral spectrum of T consists entirely of poles of the resolvent and the corresponding eigenspaces are finite-dimensional.*

PROOF. (a) $\Rightarrow$ (c): We select a free ultrafilter $\mathfrak{U}$ on $\mathbf{N}$ and embed $\mathfrak{A}$ into the $\mathfrak{U}$-product $\hat{\mathfrak{A}}$. If we define

$$\|\hat{x}\| = \lim_{\mathfrak{U}} \|x_n\|, \qquad \hat{x} \in \hat{\mathfrak{A}}, \quad (x_n) \in \hat{x},$$

$\hat{\mathfrak{A}}$ is a C^*-algebra with unit. Since the mapping $S \mapsto \hat{S}$, where $\hat{S}\hat{x} = (Sx_n)^\wedge$ for $(x_n) \in \hat{x}$, is an isometric algebra homomorphism from $\mathscr{L}(\mathfrak{A})$ into $\mathscr{L}(\hat{\mathfrak{A}})$ ([11, V.1.3]), $\hat{T}$ is uniformly ergodic on $\hat{\mathfrak{A}}$ and we have $\dim F(T) = \dim F(\hat{T})$. Moreover, $\hat{T}$ is a Schwarz mapping on $\hat{\mathfrak{A}}$. It is easy to see that the second adjoint $\hat{T}^{**}$ of $\hat{T}$ is a Schwarz mapping on the W*-algebra $\hat{\mathfrak{A}}^{**}$. Therefore the assertion follows by using Theorem 2.2, Proposition 3.1 and Proposition 3.2.

(c) $\Rightarrow$ (b): By the assumptions $\sigma(T) \cap \Gamma = \{\alpha_1, \cdots, \alpha_m\}$ for some $m \in \mathbf{N}$, and the residuum P_i of the resolvent $R(\lambda, T)$ at α_i is of finite rank. Thus $Q = \sum_{i=1}^m P_i$ is of finite rank, too. Let $T = Q + R$ where $R = T(I_{\mathfrak{A}} - Q)$. Then $r(R) < 1$ so there is $n_0 \in \mathbf{N}$ such that $\|R^{n_0}\| < 1$. But $T^{n_0} = (Q + R)^{n_0} = S + R^{n_0}$ where S is compact. Hence there exists a compact operator K on $\mathfrak{A}$ such that $\|T^{n_0} - K\| < 1$.

(b) $\Rightarrow$ (a): This follows from [3, VIII.8.4]. ∎

3.4. REMARKS. (a) For more results concerning uniformly ergodic maps on W*-algebras we refer to [5].

(b) In contrast to the commutative situation ([11, V.4.9, 5.5]) the peripheral spectrum of T in general is not a union of finite subgroups of Γ. To see this let M_n be the C^*-algebra of all $n \times n$-matrices and choose a unitary $u \in M_n$. Then for the identity preserving Schwarz operator $T := (x \mapsto uxu^*)$, $x \in M_n$, we obtain $\sigma(T) = \{\lambda\mu^* : \lambda, \mu \in \sigma(u)\}$ which may be non-cyclic.

(c) If the fixed space of T is infinite-dimensional then there may exist elements of the peripheral spectrum of T which are not poles of the resolvent $R(\lambda, T)$. To see this let, for $n \in \mathbf{N}$, A_n be the positive operator on the (commutative) C^*-algebra $\mathbf{C}^2$ represented by the matrix

$$A_n = \begin{pmatrix} 0 & 1 \\ 1 - n^{-1} & n^{-1} \end{pmatrix}.$$

If $\mathfrak{A}$ is the l^∞-product of $\mathbf{C}^2$ and if, for $x = (x_n) \in \mathfrak{A}$, $Tx := (A_n x_n)$, then T is an identity preserving Schwarz map with infinite-dimensional fixed space on the C^*-algebra $\mathfrak{A}$, 1 is pole of the resolvent $R(\lambda, T)$ and $\sigma(T) = \{1\} \cup \{-1\} \cup$

$\{-1+n^{-1}: n \in \mathbf{N}\}$. Thus -1 is not isolated in $\sigma(T)$ hence not a pole of the resolvent $R(\lambda, T)$.

(d) If we omit the assumption "$|\alpha| = 1$" in Proposition 3.2(b), then α is not necessarily a pole of the resolvent. For example, let S be an isometry with $\sigma(S) = \{\lambda \in \mathbf{C} : |\alpha| \leqq 1\}$ on a suitable Banach space F, let $E = F \oplus \mathbf{C}$ and let $T := S \oplus 0$. Then $\sigma(T) = \sigma(S)$, $\ker(\hat{T})$ is finite-dimensional in $\hat{E}$ but 0 is not a pole of the resolvent $R(\lambda, T)$.

REFERENCES

1. M.-D. Choi, *A Schwarz inequality for positive linear maps on C*-algebra*, Ill. J. Math. **18** (1974), 565–574.

2. R. Derndinger, R. Nagel and G. Palm, *13 Lectures on Ergodic Theory*, preprint, Tübingen, 1982.

3. N. Dunford and J. T. Schwartz, *Linear Operators: Part I: General Theory*, Wiley, New York, 1958.

4. U. Groh, *The peripheral point spectrum of Schwarz operator on C*-algebras*, Math. Z. **176** (1981), 311–318.

5. U. Groh, *Uniform ergodic theorems for identity preserving Schwarz maps on W*-algebras*, J. Operator Theory, to appear.

6. B. Kümmerer and R. Nagel, *Mean ergodic semigroups on W*-algebras*, Acta Sci. Math. **41** (1979), 151–159.

7. M. Lin, *On the uniform ergodic theorem*, Proc. Am. Math. Soc. **43** (1974), 337–340.

8. M. Lin, *Quasi-compactness and uniform ergodicity of Markov operators*, Ann. Inst. Henri Poincaré **11** (1975), 345–354.

9. M. Lin, *Quasi-compactness and uniform ergodicity of positive operators*, Isr. J. Math. **29** (1978), 309–311.

10. H. P. Lotz, *Quasi-compactness and uniform ergodicity of Markov operators on $C(X)$*, Math. Z. **178** (1981), 145–156.

11. H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer-Verlag, Berlin–Heidelberg–New York, 1974.

12. E. Størmer, *Positive linear maps on operator algebras*, Acta Math. **110** (1963), 233–278.

13. S. Stratila and L. Zsido, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Wells, Kent, 1979.

14. M. Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York–Heidelberg–Berlin, 1979.

15. K. Yosida, *Functional Analysis*, *4th* edn., Springer-Verlag, Berlin–Heidelberg–New York, 1974.

16. K. Yosida and S. Kakutani, *Operator theoretical treatment of Markov process and mean ergodic theorem*, Ann. of Math. **42** (1941), 188–228.

MATHEMATISCHES INSTITUT DER UNIVERSITÄT TÜBINGEN
AUF DER MORGENSTELLE 10
74 TÜBINGEN, FRG